

Remark on multi-particle observables and entangled states with constant complexity

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We show that every density matrix of an n -particle system prepared by a quantum network of constant depth is asymptotically commuting with the mean-field observables. We introduce certain pairs of hypersurfaces in the space of density matrices and give lower bounds for the depth of a network which prepares states lying outside those pairs. The measurement of an observable which is not asymptotically commuting with the mean-field observables requires a network of depth in the order of $\log n$, if one demands the measurement to *project* the state into the eigenspace of the measured observable.

I. INTRODUCTION

In the standard formulation of quantum mechanics the states are the positive operators with trace 1 on the system's Hilbert space¹. The observables are the self-adjoint operators and their eigenvalues represent the possible measurement outcomes. The question of how to design a *preparation procedure* corresponding to a given trace-one operator or a *measurement procedure* corresponding to a self-adjoint operator remained unclear for many decades. Why should such preparation or measurement procedures exist at all? Their existence seemed to be just a common *belief* of the community of physicists. Being aware of the abundance of states and observables corresponding to a many-particle quantum system, one might question this postulate. Since the old problem of Schrödinger's Cat is essentially the question of the set of states and observables of macroscopic systems [1,2], one should even accept this question as a problem of *philosophical* relevance.

However, in the context of quantum computing research and due to the recent experimental and theoretical progress in quantum optics, this question became a serious subject of research (e.g. [3–10]). The connection of the problem described above to the subject of quantum computing research can roughly be sketched as follows: Define an *ideal quantum computer* as a quantum system fulfilling the following conditions:

1. There is a physical procedure preparing one pure state $|\psi\rangle \in \mathcal{H}$, where \mathcal{H} is the system's Hilbert space.
2. There is a set of ('basic') unitary transformations acting on \mathcal{H} which can be implemented by a physical process and which are universal in the sense, that any unitary transformation can approximately be obtained by applying a sequence of basic transformations.

3. There is a read-out mechanism given by the measurement of one non-degenerated observable a acting on \mathcal{H} .

In an ideal quantum computer, every pure state can approximately be prepared by performing the appropriate unitary transformations after having prepared the state $|\psi\rangle$. Due to the evident operational meaning of convex combination on the set of density matrices, one has preparation procedures for every density matrix on \mathcal{H} . Analogously, one can find a measurement procedure for any self-adjoint operator b by writing it as $b = uf(a)u^*$ for an appropriate unitary operator u and an appropriate function f . Then b can be measured by the procedure: 'implement u , measure² a afterwards, and apply the function f to the result.'

In our opinion this framework allows a formulation of the problem of Schrödinger's Cat in a way which is more explicit than it has ever been before: On the one hand, one has strong evidence for the belief that a system being composed from many particles has *quantum and classical* aspects [11], on the other hand, if the system fulfills the axioms of the ideal quantum computer, *there is no* non-trivial observable which is compatible with all the other ones.

But the oversimplified answer 'we can measure and prepare everything' is a result of idealized assumptions. One may even state that these assumptions ignore in some sense the laws of thermodynamics: If \mathcal{H} is the state space of a many particle system, the preparation of a pure state $|\psi\rangle$ would even violate the Third Law. Furthermore, fundamental bounds on the preparation of states and the measurability might stem from the laws of quantum mechanics itself [5].

Among other things, a serious analysis has to take into account the following objections to the idealized assumptions:

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¹Here we ignore the case of a quantum system with superselection rules.

²Note that this measurement procedure does not project the state vector into the eigenspaces of b , it only reproduces the correct probabilities for the measurement outcomes. Stronger senses of measurement procedures will be considered below.

1. **The problem of complexity:** Generically, the set of basic transformations will be quite small compared to the whole set of unitary operations. Accordingly, the number of basic transformations needed for the implementation of a generic one will grow rather fast with the size of the state space \mathcal{H} . The statement ‘every unitary transformation can be implemented *in principle*’ becomes doubtful for particle numbers of macroscopic order.
2. **The problem of reliability:** We can neither expect that the preparation procedure leads to a pure initial state nor that any realizable operation acts on the density matrix like a conjugation with a unitary operator. Taking into account a finite error probability it will rather be another completely positive map. If a preparation or measurement procedure relies on a rather complex iteration of basic operations, it is a non-trivial problem to determine whether the procedure is sensitive to errors during the implementation of the basic operations. In [12], for instance, we have shown that the preparation of states showing *quantum* uncertainty on the macroscopic level in a sense explained below would require quite small error probabilities for the basic operations.

Of course one could consider these statements rather as statements about ‘practical’ problems of realization of quantum computers and question its relevance with respect to the fundamental question described in the beginning. But one should not ignore that limitations in accuracy of processes might be deeply connected with thermodynamics. In [13], for instance, we analyzed in which sense the resource requirements for preparing an (approximate) pure state grow for increasing reliability. Whether or not such thermodynamic statements on resource requirements puts fundamental restrictions to the set of accessible states and measurable observables might be answered by the future.

Having in mind the problem of Schrödinger’s Cat, we restrict our attention to mean-field observables (see [15]) of many particle systems, which are the best candidates for constituting the classical aspect of the system since they represent at least one part of the *macroscopic* level. Then it is natural to ask the following two questions:

1. How difficult is it to prepare a state showing *large* quantum uncertainty on the macroscopic level?
2. How difficult is it to measure an observable which is *strongly* incompatible with a macroscopic one?

Both question cannot really be discussed separately: If one wants to distinguish whether a strong variation of measurement outcomes for any observable stems mainly from a *classical* statistical mixture or rather from a *quantum* superposition one has to measure an observable which is incompatible with the first one. On the other

hand, a measurement of an observable being incompatible with the first one, could lead to a *quantum* superposition of macroscopically distinct states.

The question ‘how difficult is it to prepare a superposition of macroscopically distinct states?’ can be understood in various ways: It might be the question for the *reliability* of the basic operations which would be required for such preparations or measurements (this is partly answered in [12]), it can also be interpreted as the question of *complexity* of the required algorithm. This question will be focussed on here.

We will prove lower bounds for the depth of a quantum network producing macroscopic superpositions. Since these bounds are not strong, we do not claim this would be a serious restriction for the realizability, we merely consider it as an important part of pure research to determine such bounds. Furthermore it gives insights in the structure of multi-particle entanglement since we show lower bound for the complexity of the same class of highly entangled states which has been investigated in [12]. Here a quantum network is defined to be a transformation being composed by a sequence of bilocal transformations acting on only two tensor components at once. On the one hand, this is a natural assumption from the computer scientist’s point of view, since it corresponds to the decomposition of logical networks in 2-bit gates in the theory of classical electronic devices. On the other hand, even from the physicist’s point of view, it is less artificial than it may seem at first sight: If one assumes the qubits to correspond to particles, bipartite interaction represent the structure of the fundamental forces of nature.

II. MACROREALISM AND LARGE QUANTUM COMPUTERS

Let $\mathcal{H} := \otimes_{i=1}^n \mathbb{C}^l$ for arbitrary $l \in \mathbb{N}$ be the Hilbert space of n identical quantum systems. Let $(a_i)_{i \leq n}$ be a family of self-adjoint operators with the property, that every a_i acts on the i -th component of the n -fold tensor product only, i.e.

$$a_i = \underbrace{1 \otimes 1 \dots 1}_{i-1} \otimes c \otimes \underbrace{1 \otimes \dots \otimes 1}_{n-i}$$

with an arbitrary self-adjoint c acting on \mathbb{C}^l . Let $\|c\| \leq 1/2$ where $\|\cdot\|$ denotes the operator norm defined by $\|c\| := \max_{|\psi\rangle} \|c|\psi\rangle\|/\| |\psi\rangle\|$. Then we call

$$\bar{a} := \frac{1}{n} \sum_i a_i$$

the corresponding *averaging observable*. In solid state physics, the algebra generated by them is usually referred to as the algebra of *mean-field* observables [15]. We claim, that they are some of the best candidates for constituting the *macroscopic* level of the many-particle system. This might be made plausible by taking the following example:

In a system consisting of n spin-1/2 particles, the mean-magnetization is described by the observables

$$(1/2)\bar{\sigma}_x, (1/2)\bar{\sigma}_y, (1/2)\bar{\sigma}_z$$

where $\bar{\sigma}_i$ with $i = x, y, z$ is defined as

$$\bar{\sigma}_i := \sum_j \sigma_i^{(j)}$$

and $\sigma_i^{(j)}$ is the Pauli matrix σ_i acting on the j -th tensor component. Obviously the magnetization represents a physical quantity which has strong direct evidence in every day life.

Let $\|\cdot\|_{tr}$ denote the trace norm of any matrix. It is defined by $\|a\|_{tr} := \text{tr}(\sqrt{a^\dagger a})$. In [12] we argued, that a large value of the expression

$$e_\rho := \max_{\bar{a}, \|b\| \leq 1} |\text{tr}(\rho[\bar{a}, b])| = \max_{\bar{a}} \|\bar{a}, \rho\|_{tr}$$

indicates *quantum* uncertainty on the macroscopic level³. Here *large* means that the size of the expression is rather in the order of 1 than of $1/\sqrt{n}$, the latter is the case for separable states [12]. If one defines a family of hypersurfaces in the set of density matrices by

$$H_{\bar{a}, b, r} := \{\rho \mid \text{tr}(\rho[\bar{a}, b]) = r\}$$

we find that for every $r \geq 0$

$$e_\rho \leq r$$

is equivalent to

$$|\text{tr}(\rho[\bar{a}, b])| \leq r \quad \forall b \text{ with } \|b\| \leq 1, \forall \bar{a}.$$

Hence the convex set $\{\rho \mid e_\rho \leq r\}$ is enclosed by the family of pairs of hypersurfaces

$$(H_{\bar{a}, b, \pm r})_{\bar{a}, b},$$

where \bar{a} is any averaging observable and b and $\|b\| \leq 1$. In the following we will prove lower bounds for the depth of a quantum network required for crossing these hypersurfaces.

To formalize the term ‘quantum network’ we introduce the following terminology:

Definition 1 A **bilocal** or **local** map on a quantum system with Hilbert space

$$\mathcal{H} := (\mathbb{C}^l)^{\otimes n} \quad \text{with } l, n \in \mathbb{N}$$

is a completely trace preserving map on the set of density matrices on \mathcal{H} acting trivially on each tensor component except two or one, respectively, i.e., it is given by a canonical embedding of a completely positive trace preserving map acting on the density matrices on the Hilbert space $(\mathbb{C}^l)^{\otimes 2}$ or \mathbb{C}^l , respectively. The tensor components $\{i, j\} \subset \{1, \dots, n\}$ or $\{i\} \subset \{1, \dots, n\}$ on which the map acts nontrivially is called its **support**.

Definition 2 A **step** of a quantum network is a set $S := \{G_1, \dots, G_m\}$ of bilocal or local maps with mutually disjoint supports. Write $S(\rho)$ for $(G_1 \circ \dots \circ G_m)(\rho)$.

A **quantum network** of depth k is a sequence $A := (S_1, \dots, S_k)$ of k steps. Write $A(\rho)$ for $(S_k \circ \dots \circ S_1)(\rho)$. By duality, A defines a completely positive unital⁴ map A^* on the set of observables by

$$\text{tr}(A^*(a)\rho) = \text{tr}(aA(\rho)).$$

Furthermore we define:

Definition 3 The **support** of an observable a is the set $S \subset \{1, \dots, n\}$ of tensor components on which the operator a acts nontrivially.

We observe:

Lemma 1 Let a be an observable with support of size l . Let A be a quantum network of depth k . Then the support of $A^*(a)$ is $l \cdot 2^k$ at most.

Proof: By easy induction: Every step of A can double the size of the support at most. \square

Now we are able to prove one of our main statements:

Theorem 1 Let ρ be a separable state. Let A be a quantum network of depth k . Then we have

$$e_{A(\rho)} \leq \sqrt{\frac{2}{n}} 2^k$$

³Note that we take a definition slightly differing from that one introduced in [12]: Since we are dealing with the space \mathbb{C}^l on each site instead of restricting the proofs to the case of qubits, i.e. $l = 2$, it does not seem to be appropriate to restrict the one-site-observables to projections.

⁴A map G on the observables is called *unital* if one has $G(1) = 1$ for the trivial observable 1. One can show [14] that completely positive and unital implies that G is norm decreasing, i.e., $\|G(a)\| \leq \|a\|$ for every operator a .

Proof: By convexity arguments it is sufficient to prove the theorem for the case that ρ is a product state. By Lemma 1 in [12] it is sufficient to prove

$$\sqrt{\text{tr}(\bar{a}^2 A(\rho)) - (\text{tr}(\bar{a} A(\rho)))^2} \leq \frac{1}{\sqrt{2n}} 2^k$$

for every \bar{a} . We have:

$$\begin{aligned} & \text{tr}(\bar{a}^2 A(\rho)) - (\text{tr}(\bar{a} A(\rho)))^2 \\ &= \frac{1}{n^2} \sum_{ij} (\text{tr}(A^*(a_i) A^*(a_j) \rho) - \text{tr}(A^*(a_i) \rho) \text{tr}(A^*(a_j) \rho)) \\ &=: \frac{1}{n^2} \sum_{ij} g_{ij}. \end{aligned} \quad (1)$$

Let X_i be the support of $A^*(a_i)$. Since $\|A^*(a_i)\| \leq \|a_i\| \leq 1/2$ we have $|g_{ij}| \leq 1/2$. Since ρ is a product state we have $g_{ij} = 0$ for all those pairs i, j with $X_i \cap X_j = \emptyset$. Now we just have to estimate the number of those areas X_j which intersect a given area X_i . Firstly we show (by induction over k) that for every site $y \in \{1, \dots, n\}$ there are at most 2^k areas X_j with $y \in X_j$: For $k = 0$ the statement is obvious. Let B^* the dual map corresponding to a quantum network B obtained from A by adding one more step. Let W_i be the supports of $B^*(a_i)$. Assume that $y \in X_j$ for at most 2^k areas X_j . Let the additional transformation step act on the pair (y, x) where $x \in \{1, \dots, n\}$ is an arbitrary site with $x \neq y$. Then we have $y \in W_j$ only if $y \in X_j$ or $x \in X_j$. Since both sites are contained in at most 2^k areas X_j we see that y can be an element of at most 2^{k+1} areas W_j . This completes the induction.

Since every area X_i has size 2^k at most, for every X_i there are $2^k 2^k$ areas X_j at most with nonempty intersection. Hence $g_{ij} \neq 0$ for 4^k pairs (i, j) at most.⁵ Hence the term (1) is less or equal to $4^k / (2n)$. \square

The theorem can be rephrased in terms of the hypersurfaces introduced above:

Corollary 1 *Let $r > 0$. Let b with $\|b\| \leq 1$ arbitrary and \bar{a} an averaging observable. Let ρ be a separable initial state and A a quantum network such that ρ and $A(\rho)$ lie on different sides of the hypersurface*

$$H_{\bar{a}, b, r}.$$

Then the depth of A is at least

$$\frac{\ln r - \ln(\sqrt{2/n})}{\ln 2}.$$

⁵ Actually the latter statement is essentially the dual formulation of Lemma 8 in [16]. We preferred the formulation on the set of observables since it turns out to be appropriate for the proof of a theorem below.

⁶ In the terminology of [17] this is an *ideal measurement* and corresponding to every spectral projection one has realized a *passive filter*.

Note that the bound given in Theorem 1 is not too far from being tight: Take an n -qubit quantum computer with $n = 2^k$ starting with the initial state

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle^{\otimes(n-1)}.$$

Perform a controlled-not with qubit 1 as control-qubit qubit 2 as target. Than perform two controlled-not with qubit 1 and 2 as control-qubit and 3 and 4 as targets. In the r -th step one takes the qubits $1, \dots, 2^{r-1}$ as control qubits and the qubits $2^{r-1} + 1, \dots, 2^r$ as targets. After k steps one has the cat state

$$\frac{1}{\sqrt{2}}|0\rangle^{\otimes n} + |1\rangle^{\otimes n},$$

i.e., a state with $e_\rho = 1$ (see [12]: Despite the slight modification in the definition of e_ρ one can adopt the proof for the equation $e_\rho = 1$ given therein.).

Since we have answered the question of the complexity of a quantum network producing states which show large quantum uncertainty on the macroscopic level we shall focus on the second question of the complexity of measurement procedures for observables which are strongly incompatible with the averaging ones.

Firstly we should make clear, what we mean by ‘measurement procedure for an observable a ’, since this will turn out to be essential:

Definition 4 *Let a be a self-adjoint operator acting on a quantum system’s Hilbert space of finite dimension. Let $a = \sum_{i \leq j} \lambda_i P_i$ be its spectral decomposition.*

- **weak sense of measurement:**

*A procedure is said to be a **measurement procedure** for a if it has outcomes $\lambda_1, \dots, \lambda_j$ and the outcome λ_i has the probability $\text{tr}(\rho P_i)$ for a system prepared in the state ρ .*

- **strong sense of measurement:**

If the procedure changes the state in such a way that one obtains the state

$$\frac{P_i \rho P_i}{\text{tr}(\rho P_i)}$$

*in case of the result ‘ λ_i ’ we call the procedure a ‘**von-Neumann-measurement**’. Note that this assumption is much stronger than the requirement that the state vector lies in the eigenspace of the measured eigenvalue after the measurement. It even has to be **projected**.⁶*

Now we should make clear which kind of observables we assume to be measurable *directly*:

Postulate 1 *In the weak sense, one can measure every observable which is a function of an observable a of the form*

$$a = \otimes_i a_i$$

where every a_i is a self-adjoint operator acting on the i -th component of the tensor product.

Postulate 2 *In the strong sense one can only measure those observables a directly which have the property that every spectral projection P_i of a is of the form*

$$P_i = \otimes Q_j^{(i)},$$

where $Q_j^{(i)}$ is an orthogonal projection acting on the i -th tensor component.

These postulates assume, that a quite natural way of a measurement of a many-particle system is given by measuring some or all the particles separately. The set of possible outcomes is then given by the cartesian product of the sets of possible outcomes of the one-particle measurements. Of course it would also be natural to assume that one single measurement apparatus interacts with many particles in the same way. Then one would obtain rather complicated measurements as basic ones. The question ‘which observable can be measured in the *most direct* way’ is hard to answer and it is not even clear how to give a definite meaning to the term ‘most direct’. Our postulates should merely be considered as a first attempt to formalize the intuitive evidence for the fact that many-particle systems have an abundance of observables which seem to require rather sophisticated measurement procedures (in case they exist at all).

To illustrate that the difference between the weak and the strong sense of measurement is essential we take the observable

$$\otimes_i \sigma_x^{(i)},$$

which can be measured directly in the weak sense by measuring every qubit in the eigenbasis of σ_x . This observable is strongly incompatible with the averaging observable

$$\bar{\sigma}_z$$

in the sense that one has

$$\|[\bar{\sigma}_z, \otimes_i \sigma_x^{(i)}]\| = 2.$$

Easy calculation shows, that the observable

$$\otimes_i \sigma_x^{(i)}$$

is one of the best for distinguishing between the macroscopic *coherent* superposition

$$\frac{1}{2}(|0\rangle^{\otimes n} \langle 0|^{\otimes n} + |1\rangle^{\otimes n} \langle 0|^{\otimes n} + |0\rangle^{\otimes n} \langle 1|^{\otimes n} + |1\rangle^{\otimes n} \langle 1|^{\otimes n})$$

and the corresponding *mixture* of macroscopic distinct states:

$$\frac{1}{2}(|0\rangle^{\otimes n} \langle 0|^{\otimes n} + |1\rangle^{\otimes n} \langle 1|^{\otimes n}).$$

It is easy to see that our way of measuring $\otimes_i \sigma_x^{(i)}$ is far away from being a von-Neumann-measurement: Our procedure discriminates 2^n different measurement outcomes in order to measure an observable with only two different eigenvalues. A measurement in the strong sense can be designed as follows:

1. Perform a unitary transformation u such that

$$u^\dagger (\otimes_i \sigma_x^{(i)}) u = 1 \otimes 1 \otimes \dots \otimes 1 \otimes \sigma_x.$$

2. Measure the observable

$$1 \otimes 1 \otimes \dots \otimes 1 \otimes \sigma_x.$$

3. Perform the transformation u^\dagger .

Essentially, these methods for measuring a highly degenerated observable without destroying ‘too much coherence’ are necessary for quantum error correction since the computation of the error syndrome [18] must not destroy the encoded quantum information.

But why is the difference between the two ways of measurement so important for *our* main question? – Because their effects on the states with respect to the problem of Schrödinger’s Cat are in some sense even *complementary*: While the first one *destroys* macroscopic superpositions, the second one can *prepare* them. This can be seen as follows: Perform the first measurement procedure. Depending on the measurement outcomes, it will produce one of the 2^n states

$$\otimes_i |\pm x_i\rangle,$$

where $|\pm x_i\rangle$ denotes the eigenstate of the Pauli matrix σ_x with eigenvalue $+1$ or -1 at the i -th qubit. Note that for every initial state the measurement produces a product state. On the other hand, take the initial state

$$|0\rangle^{\otimes n}$$

and perform the second procedure. Since the projectors on the eigenspaces of $\otimes_i \sigma_x^{(i)}$ are given by $P_- := \frac{1}{2}(\otimes_i \sigma_x^{(i)} + 1)$ and $P_+ := \frac{1}{2}(1 - \otimes_i \sigma_x^{(i)})$ we obtain

$$\frac{1}{2}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$$

if the measurement outcome was ‘ $+1$ ’ and

$$\frac{1}{2}(-|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$$

if the outcome was ‘−1’. In both cases we get a highly entangled state showing maximal quantum uncertainty with respect to the observable $\bar{\sigma}_z$.

One might ask, whether one can go beyond the bound given by Theorem 1 if one uses additional measurements in between some steps of the quantum network. It is easy to see that this possibility is already included in Theorem 1: Every measurement procedure which can be performed *directly* can be considered as a *local* map which depends on the measurement outcome. Intermediate local transformations and the preceding or the following bilocal transformation accessing the same site can be contracted to a single bilocal map. Then the additional local maps do not change the depth at all. Roughly speaking, we have found: ‘measurements with constant complexity cannot produce that kind of highly entangled states which have large parameter e_ρ .’

The following theorem elucidates this result from another point of view by showing that observables which are measurable (in the strong sense) with very low complexity are almost compatible with the averaging ones, in the sense that their spectral projections almost commute with the former ones:

Theorem 2 *Let A be an quantum network of depth k consisting of unitary transformations. Let c be an observable which can be measured directly in the strong sense. Then we have for every spectral projection P of c :*

$$\|[\bar{a}, A^*(P)]\| \leq \frac{2^k}{\sqrt{2n}}$$

for every averaging observable \bar{a} .

Proof: Since A^* is implemented by unitary transformations we have

$$\|[\bar{a}, A^*(P)]\| = \|A(\bar{a}), P\|.$$

Set $c_i := A(a_i)$. We find

$$\left\| \sum_i c_i P - P \sum_i c_i \right\|^2 = \left\| \left(\sum_i c_i P - P \sum_i c_i \right)^2 \right\|. \quad (2)$$

We set $d_i := c_i - P c_i P$. Obviously $[d_i, P] = [c_i, P]$. Since $P d_i P = 0$ we can write expression (2) as

$$\begin{aligned} & \left\| \sum_{ij} (d_i P - P d_i) (d_j P - P d_j) \right\| \\ &= \left\| \left(\sum_i d_i P \right) \left(\sum_i d_i P \right)^\dagger + \left(\sum_i d_i P \right)^\dagger \left(\sum_i d_i P \right) \right\| \\ &\leq 2 \left\| \sum_i P d_i \right\|^2 \leq 2 \sum_{i,j} \|P d_i d_j P\|. \end{aligned} \quad (3)$$

Easy calculation shows

$$P d_i = P c_i P^\perp \quad \text{and} \quad d_j P = P^\perp c_j P.$$

Therefore we obtain

$$\left\| \left[\sum_i c_i, P \right] \right\|^2 \leq 2 \sum_{i,j} \|P c_i P^\perp c_j P\|$$

Now we argue that the sum has to be taken over those pairs (i, j) only which have the property that the supports of c_i and c_j intersect: By assumption P has the form

$$P = \otimes_i Q_i$$

where every Q_i is a projection on the i -th site. The orthogonal projection P^\perp can be split up into a direct sum in the following way: For any binary word $b \in \{0, 1\}^n$ define the projection

$$R_b := \otimes_{i \leq n} S_i$$

where $S_i = Q_i$ if the i -th digit of b is ‘0’ and $S_i = Q_i^\perp$ otherwise. Then P^\perp can be written as:

$$P^\perp = \oplus_{b \in \{0,1\}^n \setminus \{0 \dots 0\}} R_b.$$

Let $r \in \{1, \dots, n\}$ be such that b has the digit ‘1’ at the site r . If the supports of c_i and c_j are disjoint, the site r cannot be an element of both. Hence at least one of the two terms

$$P c_i R_b \quad \text{and} \quad R_b c_j P$$

vanishes, since $Q_r^\perp c_j Q_r = 0$ if r is not an element of the support of c_j .

In the proof of Theorem 1 we have shown that for every c_i there are at most 4^k operators c_j such that their supports intersect. Since $\|c_i\| \leq \|a_i\| \leq 1/2$ we have

$$\|A(\bar{a}), P\| \leq \sqrt{\frac{1}{2n} 4^k} = \sqrt{\frac{1}{2n}} 2^k.$$

□

III. CONCLUSIONS

In agreement with our approach in [12] we introduced a parameter e_ρ indicating that a many-particle state ρ shows *quantum* uncertainty on the macroscopic level. We showed that this class of highly entangled states with large e_ρ requires a quantum network with depth $\Theta(\log n)$. In [12] we could give a set of families of parallel hypersurfaces, where every family is parameterized by $r \in [-1, 1]$. For the hypersurfaces with increasing $|r|$ we could prove increasing lower bounds for the depth of a quantum network preparing states lying outside those pairs corresponding to $\pm r$. Since we consider states with large parameter e_ρ as those one constituting the philosophical problem of Schrödinger’s Cat in the debate about

macrorealism we have proven that states which are prepared by a quantum network of depth $O(1)$ are consistent with macrorealism.

In analogy, we investigated the depth of a network required for measuring those observables which constitute the conflict with macrorealism in the sense we had explained. Here we obtained the same $\Theta(\log n)$ - bound as for the preparation of states with large e_ρ .

Our results put strong restrictions on the set of entangled states which can be obtained by networks with constant depth including intermediate measurements. Further understanding of ‘very-low-complexity entanglement’ has to be left to the future. Since any tensor product structure of a state for n particles can already be destroyed by a quantum network of depth 2, this seems to be non-trivial at all.

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